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q-Index on braided non-commutative spheres

D. Gurevich[a,*], R. Leclercq[a], P. Saponov[b]

[a] *Department of Mathematics, ISTV, Université de Valenciennes, B.P. 311, 59304 Valenciennes, Cedex, France*
[b] *Theory Department, Institute for High Energy Physics, 142281 Protvino, Russia*

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Abstract

To some Yang–Baxter braidings of Hecke type we assign algebras called braided non-commutative spheres. For any such algebra, we introduce and compute a q-analog of the standard pairing $\mathrm{Ind}: K_0(A) \times K^0(A) \to \mathbb{Z}$ called a non-commutative index. Unlike the standard non-commutative index, our q-analog is based on the so-called categorical trace specific for a braided category in which the algebra in question is represented.

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1. Introduction

In K-theory there exists the well-known pairing (cf. [23]):

$$\mathrm{Ind}: K_0(A) \times K^0(A) \to \mathbb{Z}, \quad (1.1)$$

where A is a given associative algebra, $K^0(A)$ is the Grothendieck group of the monoid of its finite dimensional representations, and $K_0(A)$ is the Grothendieck group of classes

* Tel.: +33 3 2725 14 69; fax: +33 3 2751 19 00.
E-mail address: d.gurevich@neuf.fr (D. Gurevich).

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of one-sided projective A-modules.[1] (Observe that according to the Serre–Swan approach such modules are considered as appropriate analogs of vector bundles on a variety.) We will only deal with the finite dimensional representations of algebras in question taking the category of finite dimensional $U(sl(n))$-modules as a pattern. In this sense our setting is purely algebraic. This is the main difference of our approach from that based on the Connes spectral triples in which a considerable amount of functional analysis is involved (cf. [1], where the quantum function algebra $SU_q(2)$ is studied from this viewpoint).

Any one-sided projective module can be identified with an idempotent $e \in \mathrm{Mat}(A)$ where $\mathrm{Mat}(A)$ stands for the inductive limit of the algebras $\mathrm{Mat}_n(A)$ of $n \times n$ matrices with entries from A equipped with the natural embeddings $\mathrm{Mat}_n(A) \hookrightarrow \mathrm{Mat}_{n+1}(A)$.

Given a representation $\pi_U : A \to \mathrm{End}(U)$ and an idempotent $e \in \mathrm{Mat}(A)$, the pairing (1.1) is defined by

$$\mathrm{Ind}\,(e, \pi_U) = \mathrm{tr}(\pi_U(\mathrm{tr}\, e)) = \mathrm{tr}(\pi_U(e)), \tag{1.2}$$

where π_U is naturally extended to $\mathrm{Mat}(A)$. (It is not difficult to see that $\mathrm{Ind}\,(e, \pi_U)$ does not depend on a representative of a class from $K_0(A)$ or $K^0(A)$.)

In what follows the pairing (1.1) will be called *the non-commutative* (*NC*) *index*.

In this paper we introduce a "braided" version of the NC index. This version is based on the so-called *categorical trace* (see Section 2) and motivated by the "braided" nature of the algebras we shall deal with. These algebras are quotients of some braided analogs of enveloping algebras $U(gl(n))$ and $U(sl(n))$ and are thought of as braided NC counterparts of orbits in $sl(n)^*$. We calculate the braided NC index on a particular class of such type orbits.

Before introducing the algebras mentioned above let us briefly describe the braided categories in which the algebras will be represented. Any such a category is generated by a finite dimensional vector space V equipped with a map called *a braiding* (*morphism*):

$$R : V^{\otimes 2} \to V^{\otimes 2}, \tag{1.3}$$

which satisfies the quantum Yang–Baxter equation:

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}, \quad R_{12} = R \otimes \mathrm{id}, \; R_{23} = \mathrm{id} \otimes R. \tag{1.4}$$

Besides, we will suppose R to be of the Hecke type. This means that the braiding R satisfies the following *Hecke condition*:

$$(q\,\mathrm{id} - R)(q^{-1}\,\mathrm{id} + R) = 0, \quad q \in \mathbb{K}. \tag{1.5}$$

Hereafter $\mathbb{K}$ stands for the ground field (usually $\mathbb{C}$ but sometimes $\mathbb{R}$ is allowed) and the parameter $q \in \mathbb{K}$ is assumed to be generic (but $q = 1$ is permitted). The braidings of the Hecke type will be also called *Hecke symmetries*.

Let $\mathcal{C} = \mathcal{C}(V)$ be the category generated by the space V (see Section 2). The sets of its objects and categorical morphisms will be denoted, respectively, by $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{C})$. The category $U_q(sl(n)) - \mathrm{Mod}$ of all finite dimensional modules over the quantum group (QG) $U_q(sl(n))$ serves as an example of $\mathcal{C}(V)$. In this case the space V is the fundamental (vector)

[1] Throughout the paper all projective modules are supposed to be finitely generated.

module, the braiding R is the Drinfeld–Jimbo R-matrix and the categorical morphisms are linear maps commuting with the action of $U_q(sl(n))$.

Under some additional conditions on R (see Section 2) the braided categories in question are *rigid* (for the terminology the reader is referred to [2]). This means that for any $U \in \mathrm{Ob}(\mathcal{C})$ there exists $U_{\mathrm{r}}^* \in \mathrm{Ob}(\mathcal{C})$ (resp. $U_{\mathrm{l}}^* \in \mathrm{Ob}(\mathcal{C})$) for which one can define a non-degenerate pairing:

$$U \otimes U_{\mathrm{r}}^* \to \mathbb{K} \qquad (\text{resp. } U_{\mathrm{l}}^* \otimes U \to \mathbb{K})$$

and this map is a categorical morphism. The space U_{r}^* (resp. U_{l}^*) is called the *right* (*left*) *dual* space to U. Therefore, for any $U \in \mathrm{Ob}(\mathcal{C})$ the space of its right (resp. left) *internal endomorphisms*:

$$\mathrm{End}_{\mathrm{r}}(U) \stackrel{\mathrm{def}}{=} U_{\mathrm{r}}^* \otimes U, \qquad (\text{resp. } \mathrm{End}_{\mathrm{l}}(U) \stackrel{\mathrm{def}}{=} U \otimes U_{\mathrm{l}}^*) \tag{1.6}$$

is also contained in $\mathrm{Ob}(\mathcal{C})$.

Then, in $\mathcal{C}$ we define an important categorical morphism

$$\mathrm{tr}_R : \mathrm{End}_\varepsilon(U) \to \mathbb{K}, \quad \varepsilon = \mathrm{l}, \mathrm{r}$$

called *the categorical trace*. The super-trace is an example of such a categorical trace. Namely, in super-algebra and super-geometry this trace replaces the classical one. For a similar reason, dealing with a braided category, we make use of the categorical trace specific for this category.

Now, let us pass to algebras in question. Assume for a moment that $q = 1$. This means that our braiding R becomes involutive: $R^2 = \mathrm{id}$. For such a braiding there exists a natural way to define *a generalized Lie bracket*:

$$[\ ,\ ]_R : \mathrm{End}_\varepsilon(V)^{\otimes 2} \to \mathrm{End}_\varepsilon(V), \quad \varepsilon = \mathrm{l}, \mathrm{r} \tag{1.7}$$

(cf. [10] for detail). Being equipped with such a bracket, the space $\mathrm{End}_{\mathrm{l}}(V)$ (for definiteness we set $\varepsilon = \mathrm{l}$) becomes *a generalized Lie algebra*. It will be denoted $gl_R(V)$. (Note, that a similar generalized Lie bracket can be defined in $\mathrm{End}_{\mathrm{l}}(U)$ for any object $U \in \mathcal{C}(V)$.) For instance, a super-Lie algebra is a particular case of a generalized one.

Moreover, for the aforementioned categorical trace we have

$$\mathrm{tr}_R[X, Y]_R = 0, \quad X, Y \in \mathrm{End}(V) \tag{1.8}$$

and the subspace $sl_R(V)$ of all traceless elements is closed with respect to this bracket. Thus, the space $sl_R(V)$ is also *a generalized Lie algebra*. Then their enveloping algebras $U(gl_R(V))$ and $U(sl_R(V))$ can be defined by systems of quadratic-linear equations. Furthermore, they become *braided Hopf algebras*, being equipped with an appropriate coproduct, antipode and counit (cf. [25] for the definition). On generators $X \in gl_R(V)$ (or $X \in sl_R(V)$) this coproduct has the classical form:

$$\Delta(X) = X \otimes 1 + 1 \otimes X. \tag{1.9}$$

By means of the coproduct (which gives rise to a braided version of the Leibniz rule) we can construct an embedding

$$\mathrm{End}_{\mathrm{l}}(V) \to \mathrm{End}_{\mathrm{l}}(V^{\otimes m}), \quad \forall m. \tag{1.10}$$

Restricting these maps on the subspaces associated with the Young diagrams (the corresponding Young projectors can be constructed for any involutive braiding) we get a family of irreducible representations of the generalized Lie algebra $sl_R(V)$. Then considering all their direct sums we get a category of finite dimensional representations of the algebra in question similar to that of $sl(p)$-modules where $p = \mathrm{rk}(R)$ (see Section 2). The main difference between the latter category and a braided one consists in traces. When considering the algebra $U(gl_R(V))$ (or $U(sl_R(V))$) with the aforementioned category of finite dimensional representations it is natural to use the corresponding categorical trace in order to define all numerical characteristics (dimensions, indices, etc.).

It is this scheme that is realized in the present paper. However, here we deal with a more interesting (and more difficult) case of algebras and categories associated to some non-involutive Hecke symmetries. In this case it is not evident which algebras should be taken as $U(gl_R(V))$ and $U(sl_R(V))$. The matter is that a direct generalization of the bracket (1.7) to a non-involutive R leads to "enveloping algebras" which are not flat deformations of the classical ones even if R is a deformation of the usual flip. In other words, the dimensions of the homogeneous components of the corresponding graded algebra differ from their classical analogs.

Nevertheless, there exist algebras (denoted below as $\mathcal{L}_{\hbar,q}$ and $\mathcal{SL}_{\hbar,q}$) possessing good deformational properties and playing the role of braided analogs of enveloping algebras $U(gl(n))$ and $U(sl(n))$, respectively (though apparently they do not look like the usual enveloping algebras). They can be described in terms of the *modified reflection equation* (mRE). This equation can be defined for any braiding; for involutive R it leads to the aforementioned enveloping algebras $U(gl_R(V))$ and $U(sl_R(V))$. The algebras $\mathcal{L}_{\hbar,q}$ and $\mathcal{SL}_{\hbar,q}$ are presented in the next section.

In the case related to the QG $U_q(sl(n))$ (in the $U_q(sl(n))$ case for short) these algebras are one-sided $U_q(sl(n))$–modules unlike the $U_q(sl(n))$ themselves which is a two-sided $U_q(sl(n))$–module. Using the results of [24] it is possible to show that any $U_q(sl(n))$–module becomes an $\mathcal{SL}_{\hbar,q}$–one. For the Hecke symmetry coming from $U_q(sl(n))$ the corresponding category $\mathcal{C}(V)$ is the representation category of this QG and all its objects can be equipped with an action of the algebra $\mathcal{SL}_{\hbar,q}$. The family of all representations of the algebra $\mathcal{SL}_{\hbar,q}$ is, however, larger than that of $U_q(sl(n))$. We constrain ourselves to considering only finite dimensional $\mathcal{SL}_{\hbar,q}$–representations which are $U_q(sl(n))$–modules and are equivariant (covariant) with respect to the action of the QG.

But in general case we do not have such a useful tool as the QG. So, we modify the notion of equivariant representation in order to adapt it to a more general setting.

In the case $\mathrm{rk}(R) = 2$ we equip the category $\mathcal{C}(V)$ with an equivariant action of the algebra $\mathcal{SL}_{\hbar,q}$. Then we introduce *a braided NC sphere* as a quotient of this algebra (we get it by fixing a value of a quadratic braided Casimir element) and calculate the braided NC index for it. In the $U_q(sl(2))$ case the braided NC sphere is also called *the quantum NC sphere*.

The quantum sphere is close to the known Podleś sphere. However, while the Podleś sphere is $U_q(su(2))$–homogeneous space and is introduced via some reduction from its dual, our construction is defined as an appropriate quotient of the mRE algebra. As a consequence, the representation theory of the Podleś sphere developed in [27] differs drastically from that of $\mathcal{SL}_{\hbar,q}$.

Note, that the NC index on the Podleś sphere (namely, a particular case called equatorial sphere) was computed in the work [16] with the use of the representation theory from [27], the trace defined in [26] and idempotents introduced in [17]. Similar computations for other Podleś spheres were announced in [18].

In contrast, our method of constructing idempotents on the braided (in particular, quantum) NC spheres makes use of a braided version of the Cayley–Hamilton identity for some matrices from $\mathrm{Mat}(\mathcal{L}_{\hbar,q})$. This identity is a very powerful tool: it allows us to construct a family of projective modules for a big set of algebras. In particular, we get the so-called quantum NC spheres which are two-parameter deformations of the usual sphere. Setting $q = 1$ we get the standard NC (fuzzy) sphere and the idempotents on it considered earlier in [14]. Besides, in contrast with [16] we use the equivariant representation theory (similar to that of $sl(2)$) and the categorical (or q-)trace instead of the usual one.[2]

The paper is organized as follows. In the next section we define the categories and algebras we are dealing with. In Section 3 we introduce and compute the braided version of the NC index. In Section 4 we consider the quantum sphere as an example of our general construction. In Section 5 we discuss some aspects of our approach. In particular, we present a treatment of the NC index and its q-analog for the algebras in question as a quantum counterpart of the Euler characteristic of vector bundles on the usual sphere.

2. Categories $\mathcal{C}(V)$ and related algebras

We begin this section with a short description of the category $\mathcal{C} = \mathcal{C}(V)$ generated by a finite dimensional vector space V equipped with a Hecke symmetry R. This category forms a base of all our considerations, for its detailed description see [11].

Given a Hecke symmetry R, one can connect with it a “symmetric” (resp. “skew-symmetric”) algebra $\Lambda_+(V)$ (resp. $\Lambda_-(V)$) of the space V defined as the quotient

$$\Lambda_+(V) = \left\{ \frac{T(V)}{\mathrm{Im}(q\,\mathrm{id} - R)} \right\} \qquad \left(\text{resp. } \Lambda_-(V) = \left\{ \frac{T(V)}{\mathrm{Im}(q^{-1}\,\mathrm{id} + R)} \right\} \right).$$

Here $T(V)$ stands for the free tensor algebra. Let $\Lambda_\pm^k(V)$ be the homogeneous component of $\Lambda_\pm(V)$ of degree k. If there exists an integer p such that $\Lambda_-^k(V)$ is trivial for $k > p$ and $\dim(\Lambda_-^p(V)) = 1$, then R is called *an even symmetry* and p is called *the rank* of R: $p = \mathrm{rk}(R)$. Hereafter the symbol “dim” stands for the classical dimensions. In what follows all Hecke symmetries are assumed to be even.

[2] In the case $q \neq 1$, $\hbar = 0$ we get an algebra which is in a sense “q-commutative”. Our method of constructing projective modules is still valid for it. However, we do not consider q-index for this algebra. Our treatment of this “q-commutative” case is given in Section 5.

Using the Yang–Baxter equation (1.4) we can extend braiding (1.3) onto any tensor powers of V

$$R : V^{\otimes m} \otimes V^{\otimes n} \to V^{\otimes n} \otimes V^{\otimes m} \tag{2.1}$$

(as usual, we put $V^{\otimes 0} = \mathbb{K}$ and $(1 \otimes x) \triangleleft R = x \otimes 1, (x \otimes 1) \triangleleft R = 1 \otimes x, \forall x \in V^{\otimes m}$). (Hereafter the notation $x \triangleleft P$ (resp. $P \triangleright x$) stands for applying an operator P to a vector $x \in V$ so that the space V becomes a right (resp. left) module over the operator algebra.)

For an arbitrary fixed integer $m \geq 2$ we consider partitions $\lambda \vdash m$

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k), \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0, \quad \lambda_1 + \cdots + \lambda_k = m$$

(k is called *the height* of λ). There exists a natural way to assign a space V_λ (equipped with a set of embeddings $V_\lambda \hookrightarrow V^{\otimes m}$) to any partition λ. For this one should use the q-analogs of Young projectors well known in the theory of symmetric group (cf. [11]). By definition the spaces V_λ are simple objects of the category $\mathcal{C}$ (we will motivate this definition below). All other objects are the direct sums of the simple ones. Upon assuming R to be even of rank p we can confine ourselves to partitions such that $k \leq p$ since any space V_λ with λ such that $\lambda_{k+1} > 0$ vanishes.

Let us describe *the categorical morphisms* in $\mathcal{C}$. The term *categorical* emphasizes the difference among morphisms from $\mathrm{Mor}(\mathcal{C})$ and “internal endomorphisms” which are elements of $\mathrm{End}_\varepsilon(U) \in \mathrm{Ob}(\mathcal{C})$, $\varepsilon = \mathrm{r}, \mathrm{l}$.

We distinguish categorical morphisms of two kinds. Categorical morphisms of *the first kind* are the linear maps $V^{\otimes m} \to V^{\otimes m} m \geq 0$ coming from the Hecke algebra, as well as their restrictions to any object embedded into $V^{\otimes m}$. Recall, that the Hecke algebra H_m can be represented in $V^{\otimes m}$ by means of the Hecke symmetry R.

The categorical morphisms *of the second kind* arise from a procedure of canceling columns of height p in the Young diagram corresponding to a given partition λ. The procedure is as follows. Denote by v a generator of the one-dimensional space $\Lambda_-^p(V)$ (we call v *the associated determinant*). If one fixes a basis $\{x_i, 1 \leq i \leq n = \dim(V)\}$ in the space V, then v can be represented in the form

$$v = v^{i_1 \ldots i_p} x_{i_1} \otimes \cdots \otimes x_{i_p}.$$

Hereafter the summation over repeated indices is always understood. The tensor $v^{i_1 \ldots i_p}$ is one of the two structure tensors that define the matrix of the highest order antisymmetrizer (projector) $A^{(p)} : V^{\otimes p} \to \Lambda_-^p(V)$ in the basis $x_{i_1} \otimes \cdots \otimes x_{i_p}$ of $V^{\otimes p}$

$$A^{(p)}(R)_{i_1 \ldots i_p}^{j_1 \ldots j_p} = u_{i_1 \ldots i_p} v^{j_1 \ldots j_p}. \tag{2.2}$$

As one can show,$\triangleleft$ the associated determinant v possesses the property

$$(x \otimes v) \triangleleft R = (v \otimes x') \quad \text{and} \quad (v \otimes x) \triangleleft R = (x'' \otimes v) \quad \forall\, x \in V$$

where the correspondences $x \to x'$ and $x \to x''$ are some linear maps $V \to V$. Demand them to be scalar (multiple of the identity map) and equal to each other. In other words, let there exists a nonzero $a \in \mathbb{K}$ such that

$$(x \otimes v) \triangleleft R = a(v \otimes x) \quad \text{and} \quad (v \otimes x) \triangleleft R = a(x \otimes v).$$

The Hecke symmetry R satisfying such a requirement will be called *admissible*. In this case, by setting $\bar{R} = a^{-1}R$ we have

$$(x \otimes v) \triangleleft \bar{R} = (v \otimes x), \qquad (v \otimes x) \triangleleft \bar{R} = (x \otimes v). \tag{2.3}$$

The *canceling a column* is defined as a map ψ

$$\Lambda_-^p(V) \xrightarrow{\psi} \mathbb{K} : \psi(v) = 1.$$

Due to (2.3) ψ obeys the following condition

$$\bar{R} \circ (\mathrm{id} \otimes \psi) = (\psi \otimes \mathrm{id}) \circ \bar{R}. \tag{2.4}$$

The above relation is valid also for the map ψ^{-1} (inverse to ψ). By definition ψ and ψ^{-1} are the morphisms of the second kind. Any product (composition) $f \cdot g$ of morphisms of the both kinds gives *a categorical morphism* by definition. Also, any tensor product $f \otimes g$ of categorical morphisms will be a categorical morphism.

Remark 1. Condition (2.4) (in a little bit more general form) is sometimes included in the system of axioms for braided categories (factoriality condition). In [31] a braiding satisfying such a condition is called *natural*.

In what follows we assume R to be admissible. Then the condition on the height of λ can be strengthened: $k < p$. This means that we cancel all columns containing p boxes.

Let us emphasize that for an object V_λ with a *fixed* embedding $V_\lambda \hookrightarrow V^{\otimes m}$ a map $V_\lambda \to V_\lambda$ is a categorical morphism if and only if it is a scalar map. This is the reason to call these objects *simple*. Such an observation plays an important role in what follows. If a map $\xi : U \to U$, $U \in \mathrm{Ob}(\mathcal{C}(V))$ is proved to be a categorical morphism and if U does not contain isotypical components (i.e., simple objects V_λ imbedded in U in different ways) then on each simple component of U the map ξ is scalar.

The category thus introduced is a monoidal and quasitensor one whose braidings are restrictions of maps (2.1) onto the simple spaces V_λ and their direct sums. We call such a category *braided*. Note that its Grothendieck (semi)ring is isomorphic to that of $sl(p)$-modules where $p = \mathrm{rk}(R)$.

Moreover, the corresponding category $\mathcal{C}$ is rigid, that is for any $U \in \mathrm{Ob}(\mathcal{C})$ the dual spaces U_ε^*, $\varepsilon = \mathrm{r}, \mathrm{l}$ (right and left) are also contained in $\mathrm{Ob}(\mathcal{C})$. In particular, one can show that $\Lambda_-^{p-1}(V)$ is the dual (right and left) of V. This means that there exist non-degenerate pairings

$$\Lambda_-^{p-1}(V) \otimes V \to \mathbb{K} \quad \text{and} \quad V \otimes \Lambda_-^{p-1}(V) \to \mathbb{K} \tag{2.5}$$

which are categorical morphisms.

Let $\{x_\mathrm{r}^i\}$ (resp. $\{x_\mathrm{l}^i\}$) be the dual basis in the right (resp. left) dual space to V

$$\langle x_i, x_\mathrm{r}^j \rangle = \delta_i^j, \qquad \langle x_\mathrm{l}^j, x_i \rangle = \delta_i^j. \tag{2.6}$$

The dual basises $\{x_{\mathrm{r}}^i\}$ and $\{x_{\mathrm{l}}^i\}$ can be expressed in the form

$$x_{\mathrm{r}}^i = v^{a_1\ldots a_{p-1}i} x_{a_1} \otimes \cdots \otimes x_{a_{p-1}}, \qquad x_{\mathrm{l}}^i = v^{ia_1\ldots a_{p-1}} x_{a_1} \otimes \cdots \otimes x_{a_{p-1}}, \tag{2.7}$$

hence $\{x_\varepsilon^i\} \in \Lambda_-^{p-1}(V)$, $\varepsilon = \mathrm{r}, \mathrm{l}$. Now, pairings (2.6) can be explicitly constructed by means of the categorical morphism ψ. As a consequence, pairings (2.6) become categorical morphisms justifying the identification of $\Lambda_-^{p-1}(V)$ with the dual space of V (for details cf. [11]).

Introduce now a *categorical trace* which will play the central role in all our subsequent considerations. It is defined as a properly normalized categorical morphism $\mathrm{End}_\varepsilon(U) \to \mathbb{K}$. For details the reader is referred to [11] and we only briefly outline this construction.

For any admissible Hecke symmetry R there exists an operator Q that we call "inverse to R by column", i.e.,

$$R_{ia}^{jb} Q_{bk}^{al} = \delta_i^l \delta_k^j \Leftrightarrow Q_{ia}^{jb} R_{bk}^{al} = \delta_i^l \delta_k^j,$$

where R_{ia}^{jb} is the matrix of the Hecke symmetry in the basis $x_i \otimes x_j$:

$$(x_i \otimes x_j) \triangleleft R = R_{ij}^{kl} x_k \otimes x_l.$$

Consider the matrices

$$B_i^j = Q_{ai}^{aj}, \qquad C_i^j = Q_{ia}^{ja}.$$

Evidently, they satisfy

$$B_b^a R_{ai}^{bj} = \delta_i^j, \qquad R_{ia}^{jb} C_b^a = \delta_i^j. \tag{2.8}$$

Extending these matrices to any objects V_λ in a proper way we get the matrices B_λ and C_λ such that *the categorical trace* tr_R on the space $\mathrm{End}_{\mathrm{r}}(V_\lambda)$ (resp. $\mathrm{End}_{\mathrm{l}}(V_\lambda)$, see (1.6)) is defined as follows

$$\begin{aligned} &\mathrm{tr}_R X = \mathrm{tr}(B_\lambda \cdot \hat{X}), \quad \forall X \in \mathrm{End}_{\mathrm{r}}(V_\lambda) \\ &(\text{resp. } \mathrm{tr}_R Y = \mathrm{tr}(C_\lambda \cdot \hat{Y}), \quad \forall Y \in \mathrm{End}_{\mathrm{l}}(V_\lambda)). \end{aligned} \tag{2.9}$$

Here tr is the usual matrix trace and $\hat{X}$ (resp. $\hat{Y}$) is the matrix of the linear operator $V_\lambda \to V_\lambda$ corresponding to an element $X \in \mathrm{End}_{\mathrm{r}}(V_\lambda)$ (resp. $Y \in \mathrm{End}_{\mathrm{l}}(V_\lambda)$).

The matrices B_λ and C_λ are constructed in such a way that the map $X \to \mathrm{tr}_R X$ is a categorical morphism and, besides, the *categorical dimension*

$$\dim_R(U) = \mathrm{tr}_R\, \mathrm{id}_U \quad \forall U \in \mathrm{Ob}(\mathcal{C})$$

is an additive-multiplicative functional on the Grothendieck (semi)ring. Then for any simple object V_λ one gets

$$\dim_R(V_\lambda) = s_\lambda(q^{p-1}, q^{p-3}, \ldots, q^{3-p}, q^{1-p}),$$

where s_λ is the Schur function in p variables. A proof of this fact is given in [19,11] (in another setting an equivalent formula can be also found in [21]).

As a consequence of (2.7) the maps between V_{r}^* and V_{l}^* defined on basic elements by

$$x_{\mathrm{r}}^i \mapsto x_{\mathrm{l}}^a B_a^i, \qquad x_{\mathrm{l}}^i \mapsto x_{\mathrm{r}}^a C_a^i \tag{2.10}$$

belong to Mor($\mathcal{C}$). Therefore, the same is true for the maps defined via

$$1 \mapsto x_{\mathrm{l}}^a B_a^i \otimes x_i, \qquad 1 \mapsto x_i \otimes x_{\mathrm{r}}^a C_a^i$$

since they are compositions of the map ψ^{-1}

$$1 \overset{\psi^{-1}}{\mapsto} v = x_{\mathrm{r}}^i \otimes x_i = x_i \otimes x_{\mathrm{l}}^i.$$

and morphisms (2.10).

Our next aim is to introduce some associative algebras naturally connected to the categories involved. We consider these algebras as braided analogs of the enveloping algebras $U(gl(n))$ and $U(sl(n))$. Motivation will be given later.

As a starting point of our construction we introduce elements $l_i^j = x_i \otimes x_{\mathrm{r}}^j$ and form the matrix

$$L = \|l_i^j\|, \qquad 1 \le i, j \le n = \dim(V) \tag{2.11}$$

where the lower index enumerates rows and the upper one enumerates columns. Assume R to be an admissible Hecke symmetry and impose the following relations on the free algebra, generated by all the elements l_i^j:

$$RL_1RL_1 - L_1RL_1R - \hbar(RL_1 - L_1R) = 0, \quad \text{where } L_1 = L \otimes \mathrm{id},\ \hbar \in \mathbb{K} \tag{2.12}$$

or, explicitly,

$$R_{i_1i_2}^{a_1b_2} l_{a_1}^{b_1} R_{b_1b_2}^{c_1j_2} l_{c_1}^{j_1} - l_{i_1}^{a_1} R_{a_1i_2}^{b_1c_2} l_{b_1}^{c_1} R_{c_1c_2}^{j_1j_2} - \hbar(R_{i_1i_2}^{aj_2} l_a^{j_1} - l_{i_1}^a R_{ai_2}^{j_1j_2}) = 0.$$

We call this relation the *modified reflection equation* (mRE) and the corresponding algebra, $\mathcal{L}_{\hbar,q}$, *the modified reflection equation algebra*. For any Hecke symmetry R with $q \neq 1$ this algebra can be obtained from the non-modified one (corresponding to $\hbar = 0$) by a shift of generators $l_i^j \to l_i^j - a\delta_i^j\,\mathrm{id}$ with $a = \hbar(q - q^{-1})^{-1}$. This implies that for $q \neq 1$ the algebras $\mathcal{L}_{\hbar,q}$ are isomorphic for any $\hbar$ to each other. At $q = 1$ these algebras are isomorphic iff the corresponding parameters $\hbar$ are not equal to zero. The corresponding isomorphism can be obtained by rescaling the generators.

Now, consider the maps $\mathrm{Span}(l_i^j) \to \mathbb{K}$ and $\mathrm{Span}(l_i^j) \to \mathrm{Span}(l_i^j)^{\otimes k}$ defined on the basic elements as follows

$$l_i^j \mapsto \delta_i^j \quad \text{and} \quad l_i^j \mapsto l_i^{a_1} \otimes l_{a_1}^{a_2} \otimes \cdots \otimes l_{a_{k-1}}^j \tag{2.13}$$

(in a matrix form they can be written as $L \mapsto \mathrm{id}$ and $L \mapsto L^{\otimes k}$, respectively) which evidently belong to Mor($\mathcal{C}$). In what follows the matrices $L^{\otimes k}$ whose entries are considered as elements of the space $\mathcal{L}_{\hbar,q}$ (or $\mathcal{SL}_{\hbar,q}$ defined below) will be denoted L^k.

Proposition 2. *Let us set* $l_i^j \triangleright x_k = x_i B_k^j$. *Then the image of the left-hand side of* (2.12) *under this map is equal to* 0 *if we put* $\hbar = 1$. *Hence, we have a representation*

$$\pi_1 : \mathcal{L}_{\hbar,q} \to \mathrm{End}_{\mathrm{l}}(V), \quad \hbar = 1.$$

Proof. Straightforward calculations. Suffice it to apply the left-hand side of (2.12) to an arbitrary element of V and use property (2.8) of matrix B. □

In $\mathrm{End}_{\mathrm{l}}(V)$ we can choose the natural basis $h_i^j = x_i \otimes x_l^j$

$$h_i^j \triangleright x_k = \delta_k^j x_i.$$

Let us consider the map $\mathrm{Span}(l_i^j) \to \mathrm{End}_{\mathrm{l}}(V)$ defined on the basis as follows

$$l_i^j \mapsto h_i^a B_a^j.$$

It is a categorical morphism. Upon identifying the elements l_i^j and their images we can consider the set $\{l_i^j\}$ as another basis in the space $\mathrm{End}_{\mathrm{l}}(V)$. (Note, that in the basis h_i^j the representation π_1 becomes tautological: $\pi_1(h_i^j) = h_i^j$ and $\pi_1(h_i^j)$, being realized as a matrix, has the only nonzero (i, j)th entry which is equal to 1.)

It is worth emphasizing a difference between these basises. For the set $\{h_i^j\}$, the product

$$h_i^j \otimes h_k^l \to \delta_k^j h_i^l$$

is a categorical morphism while for $\{l_i^j\}$ the maps given in (2.13) are categorical morphisms.

We denote the space $\mathrm{End}_{\mathrm{l}}(V)$ by $gl_R(V)$ and now describe its subspace of traceless elements. For any rank p, the space $gl_R(V)$ decomposes into the direct sum of two simple spaces, one of them is one-dimensional. This one-dimensional component is generated by the element $h_i^i = x_i \otimes x_l^i = v$ or, equivalently, by $\mathbf{l} = C_i^j l_j^i$. The elements of other simple component of $\mathrm{End}_{\mathrm{l}}(V)$ are called *the traceless elements*. In the sequel such a space will be denoted $sl_R(V)$. Moreover, we define the algebra $\mathcal{SL}_{\hbar,q}$ as the quotient $\mathcal{SL}_{\hbar,q} = \mathcal{L}_{\hbar,q}/\{\mathbf{l}\}$.

Propositions 2, 4 and 6 below suggest a new way of constructing the representation theory of the algebras $\mathcal{L}_{\hbar,q}$ and $\mathcal{SL}_{\hbar,q}$ in the case $\mathrm{rk}(R) = 2$. In contrast with the usual method valid in the case related to the QG when the triangle decomposition of L into the product of L^+ and L^- is used, our approach works in the general setting (for arbitrary admissible Hecke symmetry).

Observe, that all representations in question are equivariant in the sense of the following definition.

Definition 3. Let A be either $\mathcal{L}_{\hbar,q}$ or $\mathcal{SL}_{\hbar,q}$ and $U \in \mathcal{C}(V)$ be an object with an associative product $U \otimes U \to U$ which is a categorical morphism (e.g., $U = \mathrm{End}_\varepsilon(W)$, $\varepsilon = \mathrm{r}, \mathrm{l}$ or a direct sum of tensor products of similar spaces). We say that a map

$$\pi_U : A \to U$$

is *an equivariant representation* if it is a representation (i.e., an algebra morphism) and its restriction to $gl_R(V)$ (resp. $sl_R(V)$) is a categorical morphism.

Our next step is to define representations of $\mathcal{L}_{\hbar,q}$ in all simple spaces V_λ. We constrain ourselves to the simplest case $\mathrm{rk}(R) = 2$. In this situation the simple objects of $\mathcal{C}$ are labelled by partitions of height 1: $\lambda = (m)$. The corresponding Young diagram has only one row of length m. For brevity, we will write $V_{(m)}$ instead of V_λ, $\lambda = (m)$. Note, that in this case $V_{(m)} = \Lambda_+^m(V)$.

Upon fixing a basis in the space $V_{(m)}$, we can identify $\mathrm{End}_\mathrm{l}(V_{(m)})$ with the matrix algebra $\mathrm{Mat}_{n_m}(\mathbb{K})$, $n_m = \dim(V_{(m)})$. The quantities n_m depend on a concrete form of the initial Hecke symmetry R. In the $U_q(sl(2))$ case we have $n_m = m + 1$ similarly to the classical ($q = 1$) case. However, for the so-called *non-quasiclassical* Hecke symmetries (cf. [14]) these quantities differ drastically from the classical ones. In general the quantity n_m is equal to the coefficient at t^m in the development of the rational function $(1 - nt + t^2)^{-1}$ in a series, where $n = \dim(V)$ and $\mathrm{rk}(R) = 2$.

Now, we pass to higher representations of the algebra $\mathcal{L}_{\hbar,q}$. In order to construct them explicitly we take into account that $V_{(m)} = \Lambda_+^m(V)$ is the image of the projection

$$P_+^m : V^{\otimes m} \to V^{\otimes m},$$

where P_+^m is the q-symmetrizer (a particular case of q-analogs of the Young projectors discussed below). Then to an arbitrary element $X \in \mathrm{End}_l(V)$ we assign the element $X_{(m)} \in \mathrm{End}_\mathrm{l}(V_{(m)})$ by the following rule

$$X_{(m)} \triangleright g = q^{1-m}[m]_q P_+^m (X_{(1)} \triangleright g), \quad \forall\, g \in \Lambda_+^m(V). \tag{2.14}$$

Here $X_{(1)} = X \otimes \mathrm{id}_{(m-1)}$ and $[m]_q = (q^m - q^{-m})/(q - q^{-1})$ is the q-analog of the integer m. Note that the map

$$\Delta_m : \mathrm{End}_\mathrm{l}(V) \to \mathrm{End}_\mathrm{l}(V_{(m)}), \qquad X \mapsto X_{(m)} \tag{2.15}$$

is a categorical morphism due to the structure of P_+^m. Composing π_1 with Δ_m we get the map

$$\pi_m : \mathcal{L}_{\hbar,q} \to \mathrm{End}_\mathrm{l}(V_{(m)}).$$

Proposition 4. *The image of the left-hand side of* (2.12) *under the map* π_m *is equal to* 0 *at* $\hbar = 1$. *So, we get a representation*

$$\pi_m : \mathcal{L}_{\hbar,q} \to \mathrm{End}_l(V_{(m)}), \qquad \hbar = 1.$$

Proof. This proposition can be deduced from the paper [30]. One of the main results of that paper consists in constructing representations $\mathcal{L}_{\hbar,q} \to \mathrm{End}(V^{\otimes m})$ through the basic representation π_1 described in Proposition 2. Let us consider the maps

$$\rho_m : \mathcal{L}_{\hbar,q} \to \mathrm{End}(V^{\otimes m}),$$

such that the matrices corresponding to the generators are defined as follows

$$\hat{\rho}^{\mathrm{t}}_m(l_i^j) = \pi^{\mathrm{t}}_1(l_i^j) \otimes I^{\otimes(m-1)} + \sum_{s=1}^{m-1} R^{-1}_{(s\to 1)}[\pi^{\mathrm{t}}_1(l_i^j) \otimes I^{\otimes(m-1)}]R^{-1}_{(1\to s)}. \tag{2.16}$$

Here I is the unity matrix, the superscript t means the matrix transposition and $R^{-1}_{(a\to b)}$ stands for the following chains of R-matrices

$$R^{-1}_{(a\to b)} = \begin{cases} R_a^{-1}R_{a+1}^{-1}\dots R_b^{-1} & a < b \\ R_a^{-1}R_{a-1}^{-1}\dots R_b^{-1} & a > b. \end{cases}$$

Then as was shown in [30] the maps ρ_m define representations of the algebra $\mathcal{L}_{\hbar,q}$.[3]

The module $V^{\otimes m}$ is reducible. It decomposes into a direct sum of irreducible submodules V_ν parameterized by Young tableaux corresponding to partitions $\nu \vdash m$. These submodules are extracted by the action of q-analogs P^m_ν of Young projectors. In the case under consideration we only need the projector P^m_+. Taking into account the property of q-symmetrizer

$$P^m_+ R_i^{-1} = R_i^{-1} P^m_+ = q^{-1} P^m_+, \quad 1 \le i \le m-1$$

and the following consequence of the definition of q-numbers

$$\sum_{s=0}^{m-1} q^{-2s} = q^{1-m}[m]_q$$

one can easily see that the map π_m coincides with $P^m_+ \hat{\rho}_m P^m_+$. This completes the proof. □

Remark 5. If $\mathrm{rk}(R) = 2$, it is easy to introduce a braided analog of the Lie bracket in the space $sl_R(V)$. Taking into account the decomposition

$$sl_R(V)^{\otimes 2} = V_{(4)} \oplus V_{(2)} \oplus V_{(0)}$$

we set $[\ ,\ ] : V_{(4)} \oplus V_{(0)} \to 0$ and require the map $[\ ,\ ] : V_{(2)} \to sl_R(V)$ to be a categorical morphism. This requirement defines the map $[\ ,\ ]$ uniquely, up to a factor. This bracket can be naturally extended to $gl_R(V)$ by the requirement $[\mathbf{l}, x] = 0$ for any $x \in gl_R(V)$. (A q-counterpart of the Lie algebra $sl(n)$ has been defined in [24].)

[3] Note, that if R is involutive and hence R^{-1} in (2.16) can be replaced by R then (2.16) becomes a direct consequence of the coproduct (1.9) and defines the embeddings (1.10). For non-involutive R the coproduct (1.9) fails but formula (2.16) remains valid.

Having introduced such a bracket, we can treat the algebra $\mathcal{SL}_{\hbar,q}$ as the universal enveloping algebra of the corresponding "q-Lie algebra" in a standard manner (the parameter $\hbar$ depends on a normalization of the bracket). Similarly, the algebra $\mathcal{L}_{\hbar,q}$ can be treated as the enveloping algebra of $gl_R(V)$.

However, we prefer to work without any "q-Lie algebra" structure. Similarly to the usual enveloping algebra, the algebra $\mathcal{SL}_{\hbar,q}$ has the following properties. It is generated by the space $sl_R(V)$ (more precisely, it is the quotient of the algebra $T(sl_R(V))$ modulo an ideal generated by some quadratic-linear terms). Moreover, its representation theory resembles that of $U(sl(2))$ and, being constructed via the maps Δ_m, is equivariant. This is the reason for considering the algebra $\mathcal{SL}_{\hbar,q}$ as a proper "braided analog" of the enveloping algebra $U(sl(2))$ (and similarly, the algebra $\mathcal{L}_{\hbar,q}$ is treated as the enveloping algebra of $gl_R(V)$).

The representations of the algebra $\mathcal{SL}_{\hbar,q}$ can be easily deduced from those of $\mathcal{L}_{\hbar,q}$. To construct them, we set

$$\mathbf{l} \triangleright x = 0, \quad \forall\, x \in V_{(m)}$$

and preserve prolongation (2.14) for the elements of the traceless component $sl_R(V)$.

Proposition 6. *Thus defined maps are representations of the algebra $\mathcal{SL}_{\hbar,q}$ with some $\hbar \neq 0$.*

We will refer to these representations of the algebra $\mathcal{SL}_{\hbar,q}$ as *sl-representations* and keep the same notation π_m for them: $\pi_m : \mathcal{SL}_{\hbar,q} \to \mathrm{End}(V_{(m)})$.

The exact value of $\hbar$ in Proposition 6 is not important. Given a representation of $\mathcal{SL}_{\hbar,q}$ with some $\hbar \neq 0$, we can get a representation with another $\hbar$ renormalizing the generators in an appropriate way. In the $U_q(sl(2))$ case this method of constructing representation theory of the algebra $\mathcal{SL}_{\hbar,q}$ was suggested in [5].

Up to now we considered the "left" representations of the algebras in question but we need also the "right" ones. Such representations are given by appropriate maps

$$\mathrm{Span}(l_i^j) \to \mathrm{End_r}(V_{(m)}^*).$$

Note, that we do not specify which dual space — right or left — is used in the formula above. In fact, it is of no importance due to (iso)morphisms (2.10). The right representation $\bar{\pi}_m$ of $\mathcal{L}_{\hbar,q}$ (and $\mathcal{SL}_{\hbar,q}$) in the space $\Lambda_+^m(V_l^*)$ is introduced in the standard way as the map $\bar{\pi}_m : \mathcal{L}_{\hbar,q} \to \mathrm{End_r}(\Lambda_+^m(V_l^*))$ given by the formula

$$\langle g \triangleleft \bar{\pi}_m(X), f\rangle = \langle g, \pi_m(X) \triangleright f\rangle \quad \text{for any} \quad f \in \Lambda_+^m(V),\; g \in \Lambda_+^m(V_l^*).$$

This construction is valid for an admissible Hecke symmetry of arbitrary rank. The case $\mathrm{rk}(R) = 2$ which we are dealing with leads to additional technical simplifications. The point is that in this case we can equip the space V with a non-degenerate bilinear form $V^{\otimes 2} \to \mathbb{K}$ which is a categorical morphism. Explicitly, such a bilinear form can be written as follows

$$\langle x_i, x_j\rangle = v_{ij}^{-1}. \tag{2.17}$$

Here $\|v_{ij}^{-1}\|$ is the matrix inverse to $\|v^{ij}\|$ which is invertible as has been shown in [10]. This form allows us to identify V with V_ε^*, $\varepsilon = \mathrm{r}, \mathrm{l}$ and, therefore, to define representations $\bar{\pi}_m : \mathcal{L}_{\hbar,q} \to \mathrm{End}_{\mathrm{r}}(V_{(m)})$ and their sl-counterparts $\bar{\pi}_m : \mathcal{SL}_{\hbar,q} \to \mathrm{End}_{\mathrm{r}}(V_{(m)})$. Similarly to all representations considered above $\bar{\pi}_m$ are equivariant.

Now, we define our main object — the braided NC sphere — as a quotient of $\mathcal{SL}_{\hbar,q}$.

Definition 7. Let $\mathrm{rk}(R) = 2$ and $\sigma \in \mathcal{SL}_{\hbar,q}$ be a nontrivial quadratic central element (e.g., take $\mathrm{Tr}_R L^2$, where Tr_R is defined in (3.9)). Fix $\alpha \in \mathbb{K}$. The quotient $\mathcal{SL}_{\hbar,q}/\{\sigma - \alpha\}$ will be called *the braided NC sphere* if $\hbar \neq 0$ and *braided q-commutative sphere* if $\hbar = 0$.

The spectral decomposition of this quotient is similar to the classical case

$$\frac{\mathcal{SL}_{\hbar,q}}{\{\sigma - \alpha\}} = \oplus_i V_{(2i)}.$$

Here the parameter α is assumed to be generic. Below we consider some polynomial identities (called Cayley–Hamilton) whose coefficients depend on α. Demanding their roots to be distinct, we get more concrete restrictions on α. In the particular case of the quantum sphere the element σ will be specified in Section 4.

3. Index via braided Casimir element

In [13] we suggested a way of constructing a family of projective modules over the RE algebra (modified or not) by means of the Cayley–Hamilton identity. As was shown in [12], the matrix L satisfying (2.12) with any even Hecke symmetry R obeys a polynomial relation

$$L^p + \sum_{i=0}^{p-1} \sigma_{p-i}(L) L^i = 0, \quad p = \mathrm{rk}(R), \tag{3.1}$$

where the coefficients $\sigma_i(L)$ belong to the center $Z(\mathcal{L}_{\hbar,q})$ of the algebra $\mathcal{L}_{\hbar,q}$. This relation is called *the Cayley–Hamilton identity*.

Let us consider the quotient algebra $\mathcal{L}_{\hbar,q}^\chi = \mathcal{L}_{\hbar,q}/\{I^\chi\}$, where $\{I^\chi\}$ is the ideal generated by the elements

$$z - \chi(z), \quad z \in Z(\mathcal{L}_{\hbar,q}) \tag{3.2}$$

and

$$\chi : Z(\mathcal{L}_{\hbar,q}) \to \mathbb{K}$$

is a character of $Z(\mathcal{L}_{\hbar,q})$. After turning to the quotient algebra $\mathcal{L}_{\hbar,q}^\chi$, the coefficients in (3.1) become numerical

$$L^p + \sum_{i=0}^{p-1} a_i L^i = 0, \quad a_i = \chi(\sigma_{p-i}(L)) \tag{3.3}$$

(in a particular case $\chi(\mathbf{l}) = 0$ we obtain a quotient of $\mathcal{SL}_{\hbar,q}$ denoted as $\mathcal{SL}_{\hbar,q}^\chi$).

Assuming the roots of the equation

$$\mu^p + \sum_{i=0}^{p-1} a_i \mu^i = 0$$

to be distinct, one can introduce p idempotents in the usual way

$$e_i = \prod_{j\neq i} \frac{L-\mu_j}{\mu_i-\mu_j}, \quad 0 \le i \le p-1. \tag{3.4}$$

If no character χ is fixed, then the roots μ_i can be treated as elements of the algebraic closure $\overline{Z(\mathcal{L}_{\hbar,q})}$ of the center $Z(\mathcal{L}_{\hbar,q})$ (or $\overline{Z(\mathcal{SL}_{\hbar,q})}$ if $\chi(\mathbf{l}) = 0$).

Besides the basic Cayley–Hamilton identity (3.1), we are interested in the so-called *derived* ones which are valid for some extensions of the matrix L [13]. A regular way of introducing these extensions can be realized via "a (split) braided Casimir element". Note that a non-braided ($q = 1$) version of split Casimir element was used in order to get characteristic identities related to Lie algebras (cf. [8] and the references therein).[4] Recently, it appeared in a close context in [22,28] where the so-called family algebras were introduced and studied.

The *braided* (*split*) *Casimir element* is defined to be

$$\mathbf{Cas} = \sum_{i,j} l_i^j \otimes h_j^i = \sum_{i,j} l_i^j \otimes l_j^k C_k^i \in \mathcal{L}_{\hbar,q} \otimes \mathcal{L}_{\hbar,q}. \tag{3.5}$$

Its crucial property is that the map

$$\mathbb{K} \to \mathcal{L}_{\hbar,q} \otimes \mathcal{L}_{\hbar,q}, \qquad 1 \mapsto \mathbf{Cas} \tag{3.6}$$

belongs to $\mathrm{Mor}(\mathcal{C})$. Therefore, **Cas** is a central element of the category $\mathcal{C}$ in the following sense

$$(\mathbf{Cas} \otimes x) \triangleleft \overline{R} = x \otimes \mathbf{Cas}, \quad \forall\, x \in U,\ \forall\, U \in \mathrm{Ob}(\mathcal{C})$$

with $\overline{R}$ defined in the previous section. For the proof it suffices to observe that the above relation is obviously valid for the unit of $\mathbb{K}$, hence, for **Cas** due (3.6) is a categorical morphism.

From now on we shall deal with the case $\mathrm{rk}(R) = 2$. In order to get the initial matrix L as well as its higher analogs from the braided Casimir element, one should replace $h_j^i \in \mathrm{End}_{\mathrm{l}}(V)$ in (3.5) with their images $\pi_m(h_j^i)$ in matrix realization.

Namely, consider the map

$$\begin{aligned}\pi_m^{(2)} &= \mathrm{id} \otimes \pi_m : \mathcal{L}_{\hbar,q} \otimes \mathcal{L}_{\hbar,q} \to \mathcal{L}_{\hbar,q} \otimes \mathrm{End}(V_{(m)}) \\ &= \mathcal{L}_{\hbar,q} \otimes \mathrm{Mat}_{n_m}(\mathbb{K}) = \mathrm{Mat}_{n_m}(\mathcal{L}_{\hbar,q})\end{aligned}$$

[4] Note that such characteristic identities were intensively studied by Adelaide school. For enveloping algebras these identities can be thought of as specializations of the CH ones (and are in fact equivalent to them). Also, we would like to mention the papers [9] where characteristic identities are obtained for QG and [20] where a version of the CH identifies is given for their dual objects ("RTT algebras").

and introduce the matrix $L_{(m)}$ in the following way

$$L^{\rm t}_{(m)} = \pi^{(2)}_m(\mathbf{Cas}),$$

where $L^{\rm t}$ stands for the transposed matrix.

Now, define the powers of the elements $\pi^{(2)}_m(\mathbf{Cas})$ as follows

$$(\pi^{(2)}_m(\mathbf{Cas}))^k = l_{i_1}^{j_1}\dots l_{i_k}^{j_k} \otimes \pi_m(h_{j_k}^{i_k})\dots\pi_m(h_{j_1}^{i_1}) \in \mathcal{L}_{\hbar,q}\otimes \mathrm{End}(V_{(m)}). \quad (3.7)$$

In particular, for $m = 1$ we have

$$(\pi^{(2)}_1(\mathbf{Cas}))^2 = l_i^a l_a^j \otimes h_j^i,$$
$$(\pi^{(2)}_1(\mathbf{Cas}))^3 = l_i^a l_a^b l_b^j \otimes h_j^i \in \mathcal{L}_{\hbar,q}\otimes \mathrm{End}(V),\ \text{etc.} \quad (3.8)$$

In more detail, the square $(\pi^{(2)}_1(\mathbf{Cas}))^2$ arises from the following chain

$$l_{i_1}^{j_1}\otimes h_{j_1}^{i_1} \mapsto l_{i_1}^{j_1}\otimes 1\otimes h_{j_1}^{i_1} \overset{(3.6)}{\mapsto} l_{i_1}^{j_1} l_{i_2}^{j_2}\otimes h_{j_2}^{i_2} h_{j_1}^{i_1} \mapsto l_{i_1}^{a} l_{a}^{j_2}\otimes h_{j_2}^{i_1}$$

and similarly for other powers.

It is easy to see that the element

$$(\pi^{(2)}_m(\mathbf{Cas}))^k \in \mathcal{L}_{\hbar,q}\otimes \mathrm{End}(V_{(m)}) = \mathcal{L}_{\hbar,q}\otimes \mathrm{Mat}_{n_m}(\mathbb{K}) = \mathrm{Mat}_{n_m}(\mathcal{L}_{\hbar,q})$$

is nothing but $(L^k_{(m)})^{\rm t}$. Remark, that though $L^{\rm t}\cdot L^{\rm t} \neq (L^2)^{\rm t}$, we apply the transposition operator to the matrix $(L_{(m)})^k$ as a whole and get $(\pi^{(2)}_m(\mathbf{Cas}))^k$. This implies that the matrix $L_{(m)}$ and the element $\pi^{(2)}_m(\mathbf{Cas})$ whose powers are defined by (3.7) satisfy the same CH identity.

Now, we want to define a map which is a q-analog of the trace $\mathrm{Mat}_{n_m}(\mathcal{L}_{\hbar,q}) \to \mathcal{L}_{\hbar,q}$. In general such a map depends on the way of realizing the algebra $\mathrm{Mat}_{n_m}(A)$ either as $\mathcal{L}_{\hbar,q}\otimes \mathrm{Mat}_{n_m}(\mathbb{K})$ or as $\mathrm{Mat}_{n_m}(\mathbb{K})\otimes\mathcal{L}_{\hbar,q}$.

We realize the algebra $\mathrm{Mat}_{n_m}(\mathcal{L}_{\hbar,q})$ as $\mathcal{L}_{\hbar,q}\otimes \mathrm{Mat}_{n_m}(\mathbb{K})$ and define a map Tr_R in the following way

$$\mathrm{Tr}_R : \mathcal{L}_{\hbar,q}\otimes \mathrm{Mat}_{n_m}(\mathbb{K}) \to \mathcal{L}_{\hbar,q}, \qquad \mathrm{Tr}_R \overset{\rm def}{=} \mathrm{id}\otimes \mathrm{tr}_R, \quad (3.9)$$

where tr_R is the categorical trace (2.9) and the space $\mathrm{Mat}_{n_m}(\mathbb{K})$ is identified with $\mathrm{End}_{\rm l}(V_{(m)})$. In particular, we have

$$\mathrm{Tr}_R L = \mathrm{Tr}_R \pi^{(2)}_1(\mathbf{Cas}) = l_i^j\otimes \mathrm{tr}_R(h_j^i) = l_i^j C_j^i = \mathbf{l}.$$

In the $U_q(sl(n))$ case the trace $\mathrm{Tr}_R L$ coincides with the *quantum trace* (cf. [7]) which plays an important role in the theory of the RE algebra.

Let us summarize the above construction once more. Given an admissible Hecke symmetry R, we introduce the category $\mathcal{C}$ as was shortly described in Section 2 and construct the morphism (categorical trace) $\mathrm{tr}_R : \mathrm{End}_l(U) \to \mathbb{K}$, $U \in \mathrm{Ob}(\mathcal{C})$ which is defined by R. Then with the category $\mathcal{C}$ we associate the algebra $\mathcal{L}_{\hbar,q}$ defined by system (2.12) and use this categorical trace in order to define the map Tr_R sending *matrices* with entries from $\mathcal{L}_{\hbar,q}$ treated as elements of $\mathcal{L}_{\hbar,q} \otimes \mathrm{End}_l(V_{(m)}) = \mathcal{L}_{\hbar,q} \otimes \mathrm{Mat}_{n_m}(\mathbb{K})$ into the algebra $\mathcal{L}_{\hbar,q}$.

As we have said above, the matrices $L_{(m)}$ also satisfy Cayley–Hamilton identities which we call the derived ones. Namely, there exists a monic polynomial $\mathcal{CH}_{(m)}(t)$ of degree $m+1$ (recall that $\mathrm{rk}(R) = 2$) whose coefficients belong to $Z(\mathcal{L}_{\hbar,q})$ such that

$$\mathcal{CH}_{(m)}(L_{(m)}) = 0, \quad m = 1, 2, \ldots \tag{3.10}$$

Pass now to the algebra $\mathcal{L}^{\chi}_{\hbar,q}$ (see (3.2)) and consider the image $\mathcal{CH}^{\chi}_{(m)}(t)$ of the polynomial $\mathcal{CH}_{(m)}(t)$ in this algebra. Relation (3.10) transforms into a corresponding relation in the algebra $\mathcal{L}^{\chi}_{\hbar,q}$:

$$\mathcal{CH}^{\chi}_{(m)}(L_{(m)}) = 0, \quad m = 1, 2, \ldots \tag{3.11}$$

where the coefficients of the polynomial $\mathcal{CH}^{\chi}_{(m)}(t)$ are numerical and the matrix $L_{(m)}$ is treated as an element of $\mathcal{L}^{\chi}_{\hbar,q} \otimes \mathrm{Mat}_{n_m}(\mathbb{K})$. An explicit form of $\mathcal{CH}^{\chi}_{(m)}(L_{(m)})$ is determined by the following proposition.

Proposition 8. *Let μ_0 and μ_1 be the roots of the polynomial $\mathcal{CH}^{\chi}_{(1)}(t)$* (3.3) (*at $p = 2$ this polynomial is quadratic*). *Then for each $m \geq 2$ the polynomial $\mathcal{CH}^{\chi}_{(m)}(t)$ is of the degree* $(m+1)$ *and its roots $\mu_i(m)$ are given by the formula*

$$q^{m-1}\mu_i(m) = q^{-i}[m-i]_q\mu_0 + q^{i-m}[i]_q\mu_1 + [i]_q[m-i]_q\hbar, \qquad i = 0, 1, \ldots, m. \tag{3.12}$$

Note, that for the standard NC (fuzzy) sphere this formula can be proved via the coproduct (1.9) in the algebra $sl(2)$ (cf. [28]). However, this method is not valid for the algebra $\mathcal{L}_{\hbar,q}$, $q \neq 1$ and the proof (3.12) becomes more complicated. Such a proof will be given in [15].

Assuming the roots $\mu_i(m)$, $0 \leq i \leq m$, of the polynomial $\mathcal{CH}^{\chi}_{(m)}(t)(m \geq 2)$ to be distinct we can introduce idempotents $e_i(m) \in \mathcal{L}^{\chi}_{\hbar,q} \otimes \mathrm{End}_l(V_{(m)})$ analogously to (3.4) (to get uniform notations, we put $e_i = e_i(1)$).

If upon fixing some $m \geq 2$ one multiplies (3.11) by $L^n_{(m)}$, $n \geq 0$, and then applies Tr_R to the resulting equalities, one obtains a recurrence for $\alpha_n(m) = \mathrm{Tr}_R L^n_{(m)}$, $n \geq 0$. The general solution for such a recurrence is of the form

$$\alpha_n(m) = \sum_{i=0}^{m} \mu_i^n(m) d_i(m),$$

where the quantities $d_i(m)$ are defined by the initial conditions, i.e., by the values $\mathrm{Tr}_R L^r_{(m)}$, $r = 0, 1, \ldots, m$. Thus, we have the following proposition.

Proposition 9. *If the roots $\mu_i(m)$ of the polynomial $\mathcal{CH}^{\chi}_{(m)}(t)$ are all distinct, then there exist $d_i(m)$ such that*

$$\mathrm{Tr}_R L^n_{(m)} = \sum_{i=0}^{m} \mu_i^n(m) d_i(m), \quad n = 0, 1, 2, \ldots$$

The coefficients $d_i(m)$ in the above expansion are functions in the roots $\mu_i(m)$. These functions are singular if there are coinciding roots. If we treat $\mu_i(m)$ as elements of $\overline{Z(\mathcal{L}_{\hbar,q})}$ (or $\overline{Z(\mathcal{SL}_{\hbar,q})}$), then the quantities $d_i(m)$ become elements of the field of fractions of the algebra $\mathcal{L}_{\hbar,q}$ (or $\mathcal{SL}_{\hbar,q}$).

Consider a representation $\overline{\pi}_k : \mathcal{L}_{\hbar,q} \to \mathrm{End_r}(V_{(k)})$ of the algebra $\mathcal{L}_{\hbar,q}$ (or $\mathcal{SL}_{\hbar,q}$) defined at the end of Section 2. It is easy to see that for a generic q the map $\overline{\pi}_k$ is surjective and hence for any $z \in Z(\mathcal{L}_{\hbar,q})$ the operator $\overline{\pi}_k(z)$ is scalar

$$\overline{\pi}_k(z) = a_k(z)\mathrm{id}, \quad a_k(z) \in \mathbb{K}, \ \forall\, z \in Z(\mathcal{L}_{\hbar,q}).$$

Therefore, we can define a character $\chi_k : Z(\mathcal{L}_{\hbar,q}) \to \mathbb{K}$ in the following way

$$\chi_k(z) = a_k(z), \quad \forall\, z \in Z(\mathcal{L}_{\hbar,q}).$$

Below we shall use the special notation for the Cayley–Hamilton polynomial in (3.11) taken at the character χ_k

$$\mathcal{CH}_{k,m}(t) = \mathcal{CH}^{\chi}_{(m)}|_{\chi=\chi_k}.$$

Also introduce another useful notation

$$L^{\mathrm{t}}_{(k,m)} = \overline{\pi}_k(l_i^j) \otimes \pi_m(h^i_j). \tag{3.13}$$

Being the image of the matrix $L^{\mathrm{t}}_{(m)}$ under the representation $\overline{\pi}_k$, the above matrix $L^{\mathrm{t}}_{(k,m)}$ is treated as an element of $\mathrm{Mat}_{n_k}(\mathrm{Mat}_{n_m}(\mathbb{K}))$. From the other hand, $L^{\mathrm{t}}_{(k,m)}$ can also be considered as an operator acting in the space $V_{(k)} \otimes V_{(m)}$. Indeed, if in (3.13) we treat $\overline{\pi}_k(l_i^j)$ and $\pi_m(h^i_j)$ as operators we get an operator acting in the space $V_{(k)} \otimes V_{(m)}$. More precisely, we put the Casimir element **Cas** between the factors $V_{(k)}$ and $V_{(m)}$ and apply it to these spaces via the representations $\overline{\pi}_k$ and π_m, respectively. This operator generated by **Cas** and acting in the product $V_{(k)} \otimes V_{(m)}$ will be denoted $\mathbf{Cas}_{(k,m)}$.

It is evident that the matrix $L_{(k,m)}$ satisfies the Cayley–Hamilton identity

$$\mathcal{CH}_{k,m}(L_{(k,m)}) = 0 \tag{3.14}$$

which is a specialization of (3.11) with $\chi = \chi_k$.

If the roots of the polynomial $\mathcal{CH}_{(k,m)}(t)$ are distinct, one can introduce idempotents $e_i(k, m)$ similarly to $e_i(m)$.

Applying the morphism

$$\mathbf{tr} = \mathrm{tr}_R^{(1)} \otimes \mathrm{tr}_R^{(2)} : \mathrm{End}_{\mathrm{r}}(V_{(k)}) \otimes \mathrm{End}_{\mathrm{l}}(V_{(m)}) \to \mathbb{K}$$

to all powers of the matrix $L_{(k,m)}$ and using the Cayley–Hamilton identity for this matrix we can prove the following proposition (similarly to Proposition 9).

Proposition 10. *Let $\mu_i(k,m)$ be all the roots of the polynomial $\mathcal{CH}_{(k,m)}(t)$. Let them be distinct. Then there exist numbers $d_i(k,m)$, $0 \le i \le m$ such that*

$$\mathbf{tr} L^n_{(k,m)} = \sum_{i=0}^{m} \mu_i(k,m)^n d_i(k,m), \quad n = 0, 1, 2, \dots$$

They are uniquely defined by the values of $\mathbf{tr}\, L^l_{(k,m)}$, $l = 0, \dots, m$.

Definition 11. The quantities $\mu_i(m)$ and $d_i(m)$ (or $\mu_i(k,m)$ and $d_i(k,m)$) will be called, respectively, *eigenvalues* and *braided* multiplicities of the matrix $L_{(m)}$ (or $L_{(k,m)}$).

Corollary 12. *Let $f(t)$ be a polynomial* (*or a convergent series*) *in t. Then*

$$\mathrm{Tr}_R f(L_{(m)}) = \sum f(\mu_i(m)) d_i(m), \qquad \mathbf{tr}\, f(L_{(k,m)}) = \sum f(\mu_i(k,m)) d_i(k,m).$$

In particular, taking as f the polynomial in the right hand side of (3.4) and its higher analogs we get the following proposition.

Proposition 13. *If the eigenvalues $\mu_i(m)$* (*resp. $\mu_i(k,m)$*) *are distinct, then*

$$\mathrm{Tr}_R e_i(m) = d_i(m), \tag{3.15}$$

$$\mathbf{tr}\, e_i(k,m) = d_i(k,m). \tag{3.16}$$

Definition 14. The quantity $\mathbf{tr}\, e_i(k,m)$ will be called the q-index and denoted $\mathrm{Ind}\,(e_i(m), \overline{\pi}_k)$.

Note that we use this term by analogy with a widely recognized term "q-trace" (which is nothing but a categorical trace corresponding to a non-involutive braiding) or "q-dimension".

Remark 15. Multiplying the trace by a factor results in a modification of the eigenvalues μ_i but does not affect the multiplicities d_i. We are only interested in the latter quantities and therefore can disregard the normalization of the trace. Similarly, the multiplicities d_i are stable under changes of the numeric factor in (2.14). However, only with the factor $q^{1-m}[m]_q$ in the definition of $X_{(m)}$ we get (3.12).

Remark 16. Similarly to [22,28], in the $U_q(sl(2))$ case we deal with elements from $(\mathcal{L}_{\hbar,q} \otimes \mathrm{End}(V_{(m)}))^{U_q(sl(2))}$, i.e., we consider $U_q(sl(2))$-invariant elements of this tensor product. Introducing a product in the family of such elements similarly to the powers of the Casimir element we get an algebra which can be considered as a q-analog of the Kirillov's family algebras (cf. [22,28]).

Taking into account that $e_i(m) \in \mathcal{L}^{\chi}_{\hbar,q} \otimes \mathrm{End}_{\mathrm{l}}(V_{(m)})$ and putting $\chi = \chi_k$ we get

$$e_i(k, m) = \bar{\pi}^{(1)}_k(e_i(m)). \tag{3.17}$$

Finally, we have

$$\mathrm{Ind}\,(e_i(m), \bar{\pi}_k) = \mathbf{tr}\, e_i(k, m) = \mathbf{tr}\bar{\pi}^{(1)}_k(e_i(m)) = \mathrm{tr}_R\, \bar{\pi}_k(\mathrm{Tr}_R\, e_i(m)). \tag{3.18}$$

This justifies our treatment of the quantity $\mathbf{tr}\, e_i(k, m)$ as a q-analog of the NC index. We would like to emphasize that (3.17) and (3.18) are valid provided that the eigenvalues $\mu_i(k, m)$ are pairwise distinct.

Remark 17. Speaking about the braided sphere, we are actually dealing with a family of such spheres depending on the value of the character $\chi = \chi_k$. So, if we treat the entries of the idempotent $e_i(m)$ as elements of $\mathcal{L}^{\chi}_{\hbar,q}$ the q-index $\mathrm{Ind}\,(e_i(m), \bar{\pi}_k)$ is well-defined only for a special value of χ depending on k.

Thus, due to (3.16) and (3.18) the computation of our q-index reduces to that of the braided multiplicity $d_i(k, m)$. Now we will show how the latter can be computed by means of the operators $\mathbf{Cas}_{(k,m)}$ defined above as images of the braided Casimir element:

$$\mathbf{Cas}_{(k,m)} : V_{(k)} \otimes V_{(m)} \to V_{(k)} \otimes V_{(m)}.$$

Since map (3.6) is a categorical morphism and the representations $\bar{\pi}_k$ and π_m are equivariant, we can conclude that each operator $\mathbf{Cas}_{(k,m)}$ belongs to $\mathrm{Mor}(\mathcal{C})$. This implies that it is scalar on any simple component of the product $V_{(k)} \otimes V_{(m)}$.

Assuming $k \geq m$ one gets the following decomposition

$$V_{(k)} \otimes V_{(m)} = V_{(k+m)} \oplus V_{(k+m-2)} \oplus \cdots \oplus V_{(k-m)}.$$

Here we use that fact that the Grothendieck (semi)ring of the category in question is isomorphic to that of $sl(2)$-modules. We compute the trace of the operator $\mathbf{Cas}_{(k,m)}$ using this decomposition. However, before doing so we would like to make the following remark.

Having fixed an object $U \in \mathrm{Ob}(\mathcal{C})$, consider an arbitrary linear operator $\mathcal{F} : U \to U$. What is its trace? The answer depends on the way this operator is realized. To $\mathcal{F}$, we can assign two elements: $F_{\mathrm{l}} \in \mathrm{End}_{\mathrm{l}}(U)$ and $F_{\mathrm{r}} \in \mathrm{End}_{\mathrm{r}}(U)$. In general, $\mathrm{tr}_R F_{\mathrm{l}} \neq \mathrm{tr}_R F_{\mathrm{r}}$. However, under the additional assumption $\mathcal{F} \in \mathrm{Mor}(\mathcal{C})$, the operator $\mathcal{F}$ should be scalar on any simple component of U. For such an operator the categorical trace is uniquely defined:

$$\mathrm{tr}_R \mathcal{F} \stackrel{\mathrm{def}}{=} \mathrm{tr}_R\, F_{\mathrm{l}} = \mathrm{tr}_R\, F_{\mathrm{r}}.$$

This follows from the trivial fact that tr_R id is the same for the right and left realization of the identity operator. To sum up, if a linear operator belongs to $\mathrm{Mor}(\mathcal{C})$, then it is scalar on simple objects and its categorical trace is uniquely defined. This observation enables us to calculate $d_i(k, m)$.

Let μ_i be the eigenvalue of $\mathbf{Cas}_{(k,m)}$ corresponding to the component $V_{(k+m-2i)}$ $(0 \le i \le m)$. Then we have

$$\mathrm{tr}_R \mathbf{Cas}^n_{(k,m)} = \sum_{i=0}^{m} \mu_i^n d_i, \quad \text{where } d_i = \dim_R(V_{(k+m-2i)}). \tag{3.19}$$

Here the eigenvalues μ_i are numbered according to decreasing spin (we use this term by analogy with the classical case: the spin of the space $V_{(i)}$ equals $i/2$). Since $\dim_R(V_{(m)}) = [m+1]_q$ in $\mathcal{C}$, we have the final result.

Proposition 18. *Let $k \ge m$ and let the eigenvalues $\mu_i(k,m)$ be all distinct. Then by arranging the eigenvalues μ_i according to decreasing spin we have*

$$\mathrm{Ind}\,(e_i(m), \bar{\pi}_k) = [m+k-2i+1]_q, \quad 0 \le i \le m. \tag{3.20}$$

Proof. Under the hypothesis formulae (3.15)–(3.18) are valid and, therefore, the family of multiplicities $d_i(k,m)$ coincides with that of d_i from (3.19). □

In the next section, we will see that if $k \ge m$ and $\bar{\pi}_k$ are sl-representations in the $U_q(sl(2))$ case, then the eigenvalues $\mu_i(k,m)$ are automatically distinct.

In the case of the standard NC (fuzzy) sphere (i.e., $q = 1$, $\hbar \ne 0$) we get

$$\mathrm{Ind}\,(e_i(m), \bar{\pi}_k) = m+k-2i+1$$

which proves the formula given in [14].

4. Example: quantum NC sphere

Let us consider a particular case of the previous construction, namely, the *quantum NC sphere*. In the framework of our general approach we will introduce it using only the corresponding Hecke symmetry, without any QG.

Let V be a two-dimensional vector space with a fixed basis $\{x_1, x_2\}$. Represent the Hecke symmetry by the following matrix

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \qquad \lambda = q - q^{-1}.$$

The matrices B and C can be computed directly and after multiplying by q^2 (which is just renormalization for the future convenience) take the form

$$B = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}, \qquad C = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}.$$

We can choose the associated determinant as $v = x_1 \otimes x_2 - qx_2 \otimes x_1$. Thus, we have

$$\|v^{ij}\| = \begin{pmatrix} v^{11} & v^{12} \\ v^{21} & v^{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}, \qquad \|v^{ij}\|^{-1} = \begin{pmatrix} 0 & -q^{-1} \\ 1 & 0 \end{pmatrix}.$$

Set

$$l_1^1 = a, \qquad l_1^2 = b, \qquad l_2^1 = c, \qquad l_2^2 = d.$$

In these generators the mRE algebra given by (2.12) takes the form

$$qab - q^{-1}ba = \hbar b, \qquad q(bc - cb) = (\lambda a - \hbar)(d - a), \qquad qca - q^{-1}ac = \hbar c,$$
$$q(cd - dc) = c(\lambda a - \hbar), \qquad ad - da = 0, \qquad q(db - bd) = (\lambda a - \hbar)b. \qquad (4.1)$$

Represent the matrix L as suggested above

$$L^{\mathrm{t}} = l_i^j \otimes \pi_1(h_j^i) = a \otimes \pi_1(h_1^1) + b \otimes \pi_1(h_2^1) + c \otimes \pi_1(h_1^2) + d \otimes \pi_1(h_2^2)$$
$$= \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

(Recall that $\pi_1(h_i^j) \triangleright x_k = \delta_k^j x_i$.) Taking into account (3.9) we find

$$\mathbf{l} = \mathrm{Tr}_R L = l_i^j C_j^i = q^{-1}a + qd.$$

It is straightforward to check that $\mathbf{l}$ is a central element in the mRE algebra. Now, let us consider the traceless component $V_{(2)} = sl_R(V)$ of the space

$$gl_R(V) = \mathrm{span}(a, b, c, d).$$

For a basis in $sl_R(V)$ we take $\{b, c, g = a - d\}$. Being reduced onto the traceless component of $gl_R(V)$, system (4.1) becomes

$$q^2 gb - bg = \hbar(q + q^{-1})b, \qquad gc - q^2 cg = -\hbar(q + q^{-1})c,$$
$$(q^2 + 1)(bc - cb) + (q^2 - 1)g^2 = \hbar(q + q^{-1})g. \qquad (4.2)$$

Let us explicitly write the vector (two-dimensional) representations of $\mathcal{SL}_{\hbar,q}$ generated by (4.2). Written respectively in the bases $\{x_1, x_2\}$ and $\{x_r^1, x_r^2\}$ the representations π_1 and $\bar{\pi}_1$ read on the generators:

$$\pi_1(g) = \kappa \begin{pmatrix} q & 0 \\ 0 & -q^{-1} \end{pmatrix}, \qquad \pi_1(b) = \kappa \begin{pmatrix} 0 & q^{-1} \\ 0 & 0 \end{pmatrix}, \qquad \pi_1(c) = \kappa \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix},$$
$$\kappa \equiv \hbar \frac{q^2 + 1}{q^4 + 1}, \qquad \bar{\pi}_1(g) = \kappa \begin{pmatrix} q & 0 \\ 0 & -q^{-1} \end{pmatrix}, \qquad \bar{\pi}_1(b) = \kappa \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix},$$

$$\bar{\pi}_1(c) = \kappa \begin{pmatrix} 0 & 0 \\ q^{-1} & 0 \end{pmatrix}.$$

In order to get the quantum NC sphere we fix a value of a nontrivial quadratic central element. As such an element we take the coefficient σ entering the Cayley–Hamilton identity (4.3) below. Then the *quantum sphere* is obtained as the quotient of algebra (4.2) modulo the ideal $\{\sigma - \alpha\}$, for some $\alpha \in \mathbb{K}$.

An explicit form of the matrices L and $L_{(2)}$ for $\mathcal{SL}_{\hbar,q}$ is as follows. Taking the sl-representation π_2 to construct $L_{(2)}$ we get

$$L = L_{(1)} = \begin{pmatrix} q[2]_q^{-1} g & b \\ c & -q^{-1}[2]_q^{-1} g \end{pmatrix},$$

$$L_{(2)} = q^{-1} \begin{pmatrix} qg & [2]_q b & 0 \\ q^{-1}c & (q-q^{-1})g & b \\ 0 & q[2]_q c & -q^{-1} g \end{pmatrix}$$

(the latter matrix is calculated in the basis $\{x_1^2, qx_1x_2 + x_2x_1, x_2^2\}$).

One can directly check that the matrix L satisfies the Cayley–Hamilton identity of the form

$$L^2 - q^{-1}\hbar L + \sigma\, \mathrm{id} = 0, \tag{4.3}$$

where

$$\sigma = -[2]_q^{-1}\, \mathrm{Tr}_R L^2 = -[2]_q^{-1}([2]_q^{-1} g^2 + q^{-1}bc + qcb) \in Z(\mathcal{SL}_{\hbar,q}). \tag{4.4}$$

The corresponding identity for the matrix $L_{(2)}$ reads

$$L_{(2)}^3 - 2\hbar \frac{[2]_q}{q^2} L_{(2)}^2 + \frac{[2]_q^2}{q^2}(q^{-2}\hbar^2 + \sigma) L_{(2)} - \hbar \frac{[2]_q^3}{q^4}\sigma = 0.$$

This was shown in [13] for a different normalization of $L_{(2)}$.

So, setting $\sigma = \alpha \in \mathbb{K}$ we come to the equation for L with numerical coefficients

$$L^2 - q^{-1}\hbar L + \alpha\, \mathrm{id} = 0$$

with the roots

$$\mu_0 = \mu_0(1) = \tfrac{1}{2}(q^{-1}\hbar - \sqrt{q^{-2}\hbar^2 - 4\alpha}),$$

$$\mu_1 = \mu_1(1) = \tfrac{1}{2}(q^{-1}\hbar + \sqrt{q^{-2}\hbar^2 - 4\alpha}).$$

The corresponding multiplicities (which coincide with $\mathrm{Tr}_R e_i(1)$ due to (3.15)) are

$$d_0(1) = \mathrm{Tr}_R e_0(1) = \mathrm{Tr}_R (L - \mu_1\, \mathrm{id})(\mu_0 - \mu_1)^{-1} = \frac{[2]_q}{2} + \frac{[2]_q \hbar}{2\sqrt{\hbar^2 - 4\alpha q^2}},$$

$$d_1(1) = \mathrm{Tr}_R e_1(1) = \mathrm{Tr}_R (L - \mu_0\, \mathrm{id})(\mu_1 - \mu_0)^{-1} = \frac{[2]_q}{2} - \frac{[2]_q \hbar}{2\sqrt{\hbar^2 - 4\alpha q^2}}.$$

As for the matrix $L_{(2)}$, its eigenvalues can be found by means of (3.12) with $m = 2$.

Our next aim is to compute the value of α corresponding to the representation $\bar{\pi}_k$ or, in other words, the value of $\chi_k(\sigma)$. Clearly, this value does not change if we replace $\bar{\pi}_k$ by π_k. Such a value (for a Casimir element being a multiple of (4.4)) was computed in [5]. Using that result we get

$$\alpha = \chi_k(\sigma) = -\frac{\hbar^2[k]_q[k+2]_q}{q^2([k+2]_q - [k]_q)^2}. \tag{4.5}$$

This implies that

$$\sqrt{q^{-2}\hbar^2 - 4\alpha} = \pm\frac{q^{-1}[2]_q[k+1]_q}{[k+2]_q - [k]_q}\hbar. \tag{4.6}$$

Choosing the positive sign in the right hand side of the formula above we get

$$\mu_0(k,1) = \frac{-q^{-1}\hbar[k]_q}{[k+2]_q - [k]_q}, \qquad \mu_1(k,1) = \frac{q^{-1}\hbar[k+2]_q}{[k+2]_q - [k]_q},$$
$$d_0(k,1) = [k+2]_q, \qquad d_1(k,1) = [k]_q.$$

Note that the eigenvalues $\mu_i(k,1)$, $i = 0, 1$, are distinct for all $k \geq 1$.

Proposition 19. *On the quantum NC sphere we have*

$$d_i(m) = \frac{q^{m-2i} + q^{-m+2i}}{2} + \frac{[m-2i]_q[2]_q\hbar q^{-1}}{2\sqrt{\hbar^2 - 4\alpha q^2}}, \quad 0 \leq i \leq m.$$

Proof. Suffice it to check that

$$\bar{\pi}_k(d_i(m))[k+1]_q = [k+m+1-2i]_q.$$

It can be easily done with the help of the following formula

$$[k+m]_q + [k-m]_q = k_q(q^m + q^{-m}). \qquad \square$$

In fact, this proposition is valid for any braided sphere since (4.5) can be shown to be true for any admissible Hecke symmetry of rank 2. Also, note that for $q = 1$ we get formula (32) from [22].

5. Concluding remarks

1. First of all, we want to emphasize that the RE algebras (especially in the $U_q(sl(n))$ case) play a very important role in integrable system theory. So, it is very interesting to study them from K-theoretical viewpoint. However, since these algebras (and their modified counterparts) are well defined for non-quasiclassical braiding as well, a natural problem arises of comparing the NC index for "quasiclassical algebras" and "non-quasiclassical ones". The crucial difference between these cases can be seen on the level of the matrices $L_{(2)}, L_{(3)}, \ldots$ and related idempotents: they depend drastically from a concrete form of the initial Hecke symmetry R. However, the resulting q-index does not depend on it. It can be explained by the properties of the categorical trace: though the matrices B and C and their extensions coming in the formulae for categorical traces depend on a concrete form of R, the categorical dimension (=categorical trace applied to the identity operator) does not (cf. [11]). Thus, essentially, we only have two cases: $q = 1$ and $q \neq 1$. The NC index corresponding to the case $q \neq 1$ is called q-index similarly to the well-recognized terms "q-trace" and "q-dimension".
2. In contrast with [27] we do not define any involution in the algebras in question. As a consequence, we do not use any $*$-operation in our representation theory either. Nevertheless, in the $U_q(sl(2))$ case it is not difficult to introduce such an involution operator $*$ in the quantum NC sphere ($\hbar$ and q are assumed to be real)

$$*b = c, \qquad *c = b, \qquad *g = g$$

   and extending it on the whole algebra via the property $*(x\,y) = (*y)\,(*x)$.

   Moreover, such an involution exists for any mRE algebra with the so-called real type R (cf. [25]). However, considered as an operator in $\mathrm{End}_\varepsilon(V)$, this involution is not a categorical morphism since the Euclidean pairing in the space V is not. (Up to a factor, the only pairing in V which is a categorical morphism is given by (2.17).) So, such an involution is somewhat useless for constructing an equivariant representation theory (also, cf. [5] for a discussion). Emphasize a very important property of our representation theory: in the $U_q(sl(2))$ case at $q \to 1$ we get $SL(2)$-equivariant representation theory.

   There is another inconvenience of the involution under discussion. It does not allow to get the quantum sphere as an $\mathbb{R}$-algebra. We refer the reader to [3] where the equatorial Podleś sphere is explicitly presented by a system of equations containing complex numbers. Consequently, the corresponding quotient algebra cannot be considered as an $\mathbb{R}$-one in contrast with the coordinate ring of the usual sphere or the NC (fuzzy) one (cf. [14]).
3. Point out a crucial property of the pairing (1.1): it is in some sense "equivariant". This means that all maps in the chain

$$\mathcal{L}_{\hbar,q} \otimes \mathrm{End}(V_{(m)}) \stackrel{\mathrm{Tr}_R}{\to} \mathcal{L}_{\hbar,q} \stackrel{\bar{\pi}_k}{\to} \mathrm{End}(V_{(k)}) \stackrel{\mathrm{tr}_R}{\to} \mathbb{K}$$

   are categorical morphisms (in the $U_q(sl(2))$ case they commute with the $U_q(sl(2))$ action). Emphasize that in the $U_q(sl(2))$ case the QG $U_q(sl(2))$ acts on both factors in $\mathcal{L}_{\hbar,q} \otimes$

$\mathrm{End}(V_{(m)})$ via the QG coproduct. Thus, the idempotents $e_i(m)$, being realized as elements of $\mathcal{L}_{\hbar,q} \otimes \mathrm{End}(V_{(m)})$, become invariant (see Remark 16). Namely, the fact that the space $\mathrm{End}(V_{(m)})$ is equipped with an action of this QG urges us to apply the *categorical* trace to the second factor of $\mathcal{L}_{\hbar,q} \otimes \mathrm{End}(V_{(m)})$ product instead of the usual one. By contrast, in [16] there was not defined any action of $U_q(sl(2))$ on the corresponding idempotents as a whole.

4. Restricting ourselves to the $U_q(sl(2))$ case, let us discuss a geometrical meaning of the NC index and its q-analog. Setting $n = m - 2i$ we can represent (3.20) in the form

$$\mathrm{Ind}\,(e_i(m), \overline{\pi}_k) = [n + k + 1]_q. \tag{5.1}$$

Thus, the q-index depends only on n and k and therefore the idempotents $e_i(m)$ and $e_{i+1}(m+2)$ give rise to the same q-index, being paired with any representation $\overline{\pi}_k$ (k must be sufficiently large).

Remark, that in the classical limit ($q = 1, \hbar = 0$) the idempotents $e_i(m)$ and $e_{i+1}(m+2)$ are equivalent (for the definition cf. [29]) and therefore belong to the same class in K_0. We do not know whether these idempotents are equivalent in the conventional sense for the quantum algebras. (The problem of their equivalence seems to be difficult.) However, as follows from Proposition 19.

$$\mathrm{Tr}_R e_i(m) = \mathrm{Tr}_R e_{i+1}(m+2), \quad 0 \le i \le m,\ \forall\, m \ge 0. \tag{5.2}$$

The modules for which the corresponding idempotents have equal traces Tr_R will be called *trace-equivalent*. It is easy to see that, for a generic q, the modules from the sequence

$$e(0), e_0(m), e_m(m), \quad m = 1, 2, \ldots$$

are not trace-equivalent. Thus, the set of classes of trace-equivalent modules is labelled by $n = m - 2i \in \mathbb{Z}$. This looks like the Picard group of the usual sphere. This gives one more reason to use the categorical trace since if one replaced Tr_R by the usual trace, then (5.2) would be wrong.

Observe, that for the usual sphere all irreducible representations of its coordinate ring are one-dimensional and the pairing of such a representation with any idempotent $e_i(m)$ is always equal to 1. Thus, NC index for line bundles on the usual sphere is meaningless. Instead, we consider the Euler characteristic of line bundles on it. In order to do so we realize the usual sphere as a complex projective variety and consider the holomorphic line bundles $\mathcal{O}(n)$, $n \in \mathbb{Z}$. Then the bundles $\mathcal{O}(n)$ and $\mathcal{O}(-n)$, $n \ge 0$, become analogs of our modules corresponding to the idempotents $e_0(n)$, $e_n(n)$. Which line bundle corresponds to which projective module depends on the complex structure on the sphere (in our setting the result depends on the sign of the root in (4.6)).

The Euler characteristic of the bundle $\mathcal{O}(n)$ is defined to be

$$\chi(\mathcal{O}(n)) = \dim\ H^0(\mathcal{O}(n)) - \dim\ H^1(\mathcal{O}(n))$$

(for $n \ge 0$ it gives the dimension of the space of global holomorphic sections). Due to the Riemann–Roch theorem we have $\chi(\mathcal{O}(n)) = n + 1$ which coincides with the above quantity $[n+k+1]_q$ at $k=0$ and $q=1$. So, for $q=1$ we consider the index (5.1) as a NC analog of the Euler characteristic of the class of the idempotents $e_i(m)$ with $n=m-2i$. However, in contrast with the commutative case this NC index depends on two arguments: n labelling classes of trace-equivalent idempotents and k labelling classes from K_0.

Similarly, for $q \neq 1$ and $\hbar \neq 0$ the q-index depends on two parameters and is given by q-counterpart of the integer $n+k+1$. At $\hbar = 0$ the algebra $\mathcal{L}_{\hbar,q}$ does not have a meaning of an enveloping algebra and we consider neither its representations nor q-index for it. In this case we take the usual sphere as a pattern. So, we consider the quantity $[n+1]_q$, i.e., the specialization of the q-index at $k=0$, as a q-analog of the Euler characteristic. To be more precise, we should call the quantity (5.1) the NC q-index considering its specialization at $k=0$ as "commutative q-index" or "q-Euler characteristic" of the q-commutative sphere (see Definition 7).

5. The scheme presented in this paper is valid for NC q-analogs of semisimple orbits (i.e., orbits of semisimple elements) in $sl(n)^*$ which are not necessarily generic ones. These analogs arising from Hecke symmetries of higher rank can be defined (at least for the $U_q(sl(n))$ case) by methods of the paper [4]. Thus, the "easy part", namely, the fact that the q-index is nothing but a q-dimension of a component in some tensor product, can be straightforwardly generalized. The proof of an analog (3.12) is, however, much harder. Nevertheless, our low dimensional computations make the following conjecture very plausible.

Conjecture 20. *Let $\mu_i 1 \le i \le p$ be roots of the polynomial $\mathcal{CH}^{\chi}_{(1)}(t)$ (3.3). Then for $\forall\, m \ge 2$ the degree of the polynomial $\mathcal{CH}^{\chi}_{(m)}(t)$ reads*

$$\deg(\mathcal{CH}^{\chi}_{(m)}(t)) = \binom{m+p-1}{m}$$

and its roots are given by the formula

$$q^{m-1}\mu_{k_1\dots k_p}(m) = \sum_{i=1}^{p} \frac{[k_i]_q}{q^{m-k_i}}\mu_i + \xi_p(k_1,\dots,k_p)\hbar, \qquad k_i \ge 0,\ k_1+\dots+k_p = m, \tag{5.3}$$

where $\xi_p(k_1,\dots,k_p)$ is the symmetric function defined as follows

$$\xi_p(k_1,\dots,k_p) = \sum_{s=2}^{p} q^{k_1+k_2+\dots+k_s-m}[k_s]_q[k_1+k_2+\dots+k_{s-1}]_q.$$

6. It is worth emphasizing again, that in the $U_q(sl(2))$ case we have a two parameter deformation of the usual sphere (more precisely, of its complexification). Let us discuss the classical analog of this two parameter family. More generally, we consider Poisson structures on any semisimple orbit in $sl(n)^*$ (or $su(n)^*$). On such an orbit there exists a family of the so-called Poisson-Lie structures (cf. [6]). Their quantization (in general,

formal) leads to algebras covariant with respect to $U_q(sl(n))$. But in this family only one bracket (up to a numerical factor) is compatible with the Kirillov one. Namely, the simultaneous quantization of the corresponding "Poisson pencil" gives rise to the quantum algebras which are appropriate quotients of $\mathcal{SL}_{\hbar,q}$ (the reader is referred to [13] for detail).

However, the properties of quantum algebras arising from the Kirillov bracket alone and those arising from the above pencil are different. The Kirillov structure is symplectic and for it there exists an invariant (Liouville) measure. It gives rise to the classical trace in the corresponding quantum algebra. On the contrary, the other brackets from the Poisson pencil are not symplectic and they have no invariant measure. Their quantization leads to the algebras with trace but this trace is braided. It is just these algebras and their "non-quasiclassical" analogs (in a particular case $\mathrm{rk}(R) = 2$) which are the main objects of the present paper.

Also note, that the Poisson-Lie structures non-compatible with the Kirillov bracket give rise to one-parameter quantum algebras. It seems that for these algebras there is no reasonable way to construct meaningful projective modules. On the usual sphere such structures do not exist due to its low dimension.

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